A New Approach For Modelling and Pricing Correlation Swaps

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Introduction

This report carries forward an earlier work (2005) on the arbitrage pricing of correlation swaps on a stock index. The theoretical derivations have been made more rigorous, and we also include tentative parameter estimates based on a break-even historical analysis. Our aim here is to provide some elementary fundamental and practical results, which may serve as guiding principles and rules of thumbs for a new category of derivatives on realised volatility and correlation. These 'statistical derivatives' have grown in popularity over the past few years, giving sophisticated investors the opportunity to take advantage of specific market structural imbalances.

Correlation swaps are over-the-counter derivative instruments allowing to trade the observed correlation between the returns of several assets, against a pre-agreed price. In the equity derivatives sphere, these contracts appeared in the early 2000's as a means to hedge the parametric risk exposure of exotic desks to changes in correlation. Exotic derivatives indeed frequently involve multiple assets, and their valuation requires a correlation matrix as input parameter. Unlike volatility, whose implied levels have become observable due to the development of listed option markets, implied correlation coefficients are unobservable, which makes the pricing of correlation swaps a perfect example of 'chicken-egg problem.' We show how a correlation swap on the constituent stocks of an index can be viewed as a simple derivative on two types of tradable variance – the square of volatility –, and derive an analytical formula for its fair value relying upon dynamic trading of these instruments.

The report is organised as follows. Section 1 gives precise definitions of the concepts of realised and implied volatility of an index and its constituent stocks, realised and implied dispersion as well as realised and implied correlation; some key mathematical properties and practical applications are then introduced. Section 2 proposes a one-factor 'toy model' for derivatives on realised variance which is a straightforward modification of the Black-Scholes (1973) model; an analytical formula for the fair value of volatility (as opposed to variance) is then derived and used for parameter estimation. Section 3 extends the toy model to two factors in order to derive an analytical formula for the fair value of realised correlation; our numerical results suggest that the fair value of a correlation swap should be close to implied correlation; finally, a formal link between dispersion trading and the hedging strategy for correlation is established.

1. Fundamentals of index volatility, constituent volatility, correlation and dispersion

1.1. Definitions

Consider a universe of N stocks $S = (S_i)_{i=1..N}$, and a vector of positive real numbers $w = (w_i)_{i=1..N}$ such that[1] $\Sigma_{i=1..N} \, w_i = 1$. Denote $S_i(t)$ the price of stock S_i at time t, with convention $S(0) = \mathbf{1}$, and define their geometric average as:

$$(\mathbf{110}) \quad I(t) \equiv \prod_{i=1}^{N} \big(S_i(t)\big)^{w_i}$$

From an econometric point of view, (S, w, I) is a simplified system[2] for the calculation of a stock index I with constituent stocks S and weights w.

We complete the quantitative setup by considering a probability space (Ω, E, P) with a P-filtration F, and assuming that the vector S of stock prices is an F-adapted, positive Ito process.

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[1] The positivity and unit sum conditions imply $N \geq 2$.

[2] In practice, most stock indexes are defined as an arithmetic weighted average, with weights corresponding to market capitalisations; as such, weights change continuously with stock prices. Additionally the constituent stocks are typically reviewed on a quarterly or annual basis.

Fundamentals of index volatility, constituent volatility, correlation and dispersion

Given a time period[3] τ and a positive Ito process X, define:

(**111**) $|\tau| \equiv \int_\tau ds$

(**112**) $\sigma^X(\tau) \equiv \sqrt{|\tau|^{-1} \int_\tau (d \ln X_s)^2}$

(**113**) $\bar{\sigma}^S(\tau) \equiv \sqrt{\sum_{i=1}^N w_i \left(\sigma^{S_i}(\tau)\right)^2}$

(**114**) $\varepsilon(\tau) \equiv \sqrt{\sum_{i=1}^N w_i^2 \left(\sigma^{S_i}(\tau)\right)^2}$

From an econometric point of view, (111) is the length of a time period, (112) is the continuously sampled realised volatility of any positive Ito process X (in particular that of the constituent stock prices S, and the index value I), (113) is the continuously sampled average realised volatility of the constituent stocks[4], and (114) corresponds to a realised residual quantity useful below.

It is easy to see that $\bar{\sigma}^S(\tau) \geq \sigma^I(\tau)$, and $\bar{\sigma}^S(\tau) > \varepsilon(\tau)$. Define:

(**115**) $d(\tau) \equiv \sqrt{\left(\bar{\sigma}^S(\tau)\right)^2 - \left(\sigma^I(\tau)\right)^2}$

(**116**) $\rho(\tau) \equiv \dfrac{\left(\sigma^I(\tau)\right)^2 - \left(\varepsilon(\tau)\right)^2}{\left(\bar{\sigma}^S(\tau)\right)^2 - \left(\varepsilon(\tau)\right)^2} \leq 1$

From an econometric point of view, (115) is the continuously sampled average realised dispersion[5] between constituent stocks, and (116) is their continuously sampled average realised correlation[6]. (116) is consistent with usual econometric and market practices (see for instance Skintzi-Refenes[7], 2005), and we refer to it as canonical realised correlation.

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[3] Here a time period may be a segment such as $[t_0, t_1]$, or a finite reunion of segments.

[4] Note that (113) corresponds to the canonical quadratic norm, whereas constituent volatility is more frequently defined as the weighted arithmetic average of realised stock volatilities. Our choice is motivated by the economic fact that only variance has a liquid market.

[5] To see this clearly rewrite: $d(\tau)^2 = |\tau|^{-1} \int_{t \in \tau} \sum_{i=1}^N w_i \left(d \ln S_i(t) - \sum_{k=1}^N w_k d \ln S_k(t) \right)^2$

[6] To see this clearly observe that: $\rho \leq \dfrac{(\sigma^I)^2 - \varepsilon^2}{\left(\sum_{i=1}^N w_i \sigma^{S_i}\right)^2 - \varepsilon^2} = \dfrac{\sum_{i \neq j} \alpha_i \alpha_j \rho^{S_i, S_j}}{\sum_{i \neq j} \alpha_i \alpha_j}$, where $\alpha_i \equiv w_i \sigma_i$ and

$\rho^{S_i, S_j}(\tau) \equiv |\tau|^{-1} (\sigma^{S_i} \sigma^{S_j})^{-1} \int_\tau (d \ln S_i)(d \ln S_j)$.

[7] Note that Skintzi-Refenes implicitly define $\bar{\sigma}^S(\tau)$ as the weighted arithmetic average of constituent volatilities.

Fundamentals of index volatility, constituent volatility, correlation and dispersion

We now consider a variance market on (S, w, I) where at any point in time t agents can buy and sell future realised variance[8] σ^2 over any given period τ, against payment at maturity[9] of a pre-agreed price[10] called implied variance[11] and denoted σ^{*2}. In the absence of arbitrage, this means that there exists a P-equivalent, F-adapted measure P^* such that for $X = I$ or $X = S_i$:

$$\textbf{(117)} \quad \sigma_t^{*X}(\tau) = \sqrt{E_t^*\left[\left(\sigma^X(\tau)\right)^2\right]}$$

where E_t^* denotes conditional expectation under P^* with respect to F_t.

Additionally, define implied constituent volatility and implied residual as:

$$\textbf{(118)} \quad \overline{\sigma}_t^{*S}(\tau) \equiv \sqrt{\sum_{i=1}^{N} w_i \left(\sigma_t^{*S_i}(\tau)\right)^2} = \sqrt{E_t^*\left[\left(\overline{\sigma}^S(\tau)\right)^2\right]}$$

$$\textbf{(119)} \quad \varepsilon_t^*(\tau) \equiv \sqrt{\sum_{i=1}^{N} w_i^2 \left(\sigma_t^{*S_i}(\tau)\right)^2} = \sqrt{E_t^*\left[\left(\varepsilon(\tau)\right)^2\right]}$$

From an economic point of view, (118) and (119) correspond to the unique no-arbitrage price of a portfolio of the N future realised variances of the constituent stocks, with weights w and $(w_i^2)_{i=1..N}$, respectively.

No arbitrage considerations imply that $\overline{\sigma}_t^{*S}(\tau) \geq \sigma_t^{*I}(\tau)$, and $\overline{\sigma}_t^{*S}(\tau) > \varepsilon_t^*(\tau)$. Define:

$$\textbf{(120)} \quad d_t^*(\tau) \equiv \sqrt{\left(\overline{\sigma}_t^{*S}(\tau)\right)^2 - \left(\sigma_t^{*I}(\tau)\right)^2} = \sqrt{E_t^*\left[\left(d(\tau)\right)^2\right]}$$

$$\textbf{(121)} \quad \rho_t^*(\tau) \equiv \frac{\left(\sigma_t^{*I}(\tau)\right)^2 - \left(\varepsilon_t^*(\tau)\right)^2}{\left(\overline{\sigma}_t^{*S}(\tau)\right)^2 - \left(\varepsilon_t^*(\tau)\right)^2} \leq 1$$

We refer to (120) as implied dispersion and (121) as canonical implied correlation.

Here, we must emphasise that, while (120) corresponds to the no-arbitrage price of realised dispersion as defined in (115), (121) does not necessarily correspond to the fair value of realised correlation as defined in (116): in general, $\rho_t^*(\tau) \neq E_t^*(\rho(\tau))$. It is the aim of Section 3 to bridge the gap between implied correlation and the fair value of future realised correlation.

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[8] Throughout this report, variance means the square of volatility, and volatility means the square root of variance.

[9] Here maturity means sup τ.

[10] In this report, the prices or values of all derivatives are considered in their natural currency and as of maturity, i.e. they are forward prices or values.

[11] Note that in the absence of a variance market, the price of variance can be determined using listed option prices (see e.g. Derman et al., 1999).

Fundamentals of index volatility, constituent volatility, correlation and dispersion

1.2. Proxy formulas for realised and implied correlation

In Appendix A, we derive proxy formulas for (116) and (121) using limit arguments. Subject to fairly reasonable conditions on the weights and the pair-wise realised correlations between constituent stocks, we find that residual terms $\varepsilon(\tau)$ and $\varepsilon_t^*(\tau)$ vanish as N goes to infinity:

$$\rho(\tau) \xrightarrow[N \to +\infty]{} \left(\frac{\sigma^I(\tau)}{\overline{\sigma}^S(\tau)} \right) \equiv \hat{\rho}(\tau)$$

$$\rho_t^*(\tau) \xrightarrow[N \to +\infty]{} \left(\frac{\sigma_t^{*I}(\tau)}{\overline{\sigma}_t^{*S}(\tau)} \right) \equiv \hat{\rho}_t^*(\tau)$$

We refer to $\hat{\rho}(\tau)$ and $\hat{\rho}_t^*(\tau)$ as realised and implied correlations, respectively.

In Exhibit 1.2.1, we compare three measures of the realised correlation of the Dow Jones EuroStoxx 50 index, over 1-month and 24-month rolling periods since 2000. We can see that the distance between the proxy and canonical measures does not exceed a few correlation points, and also that the distance between the proxy and average pair-wise measures can occasionally be significant (more than 10 correlation points), particularly for 24-month rolling periods.

In Exhibit 1.2.2, we introduce an alternative measure which is based on a reconstitution of index values with constant weights and constituent stocks at the start of each 24-month period, as well as a substitute calculation of realised constituent volatility. We can see that the distance between this measure and the average pair-wise measure does not exceed a few correlation points.

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Exhibit 1.2.1 — 1-month and 24-month realised correlation measures of the Dow Jones EuroStoxx 50 index (2000—2007)

In the two charts below, we report on each monthly listed expiry date the correlation level realised over the following (a) month or (b) 24 months, using formulas below. On each monthly start date, we retrieved the constituent stocks and their weights, and held them constant over the 1-month or 24-month time period. In all cases, realised volatilities were calculated as the annualised zero-mean standard deviation of daily log-returns, using closing levels. For index volatility, we used actual index closing levels as disseminated by the calculation agent.

$$\text{Proxy:} \left(\frac{\sigma^I}{\overline{\sigma}^S} \right)^2, \text{Canonical:} \frac{\left(\sigma^I\right)^2 - \varepsilon^2}{\left(\overline{\sigma}^S\right)^2 - \varepsilon^2}, \text{Average pair-wise:} \left(\sum_{i<j} w_i w_j \right)^{-1} \sum_{i<j} w_i w_j \rho^{S_i, S_j}.$$

(a)
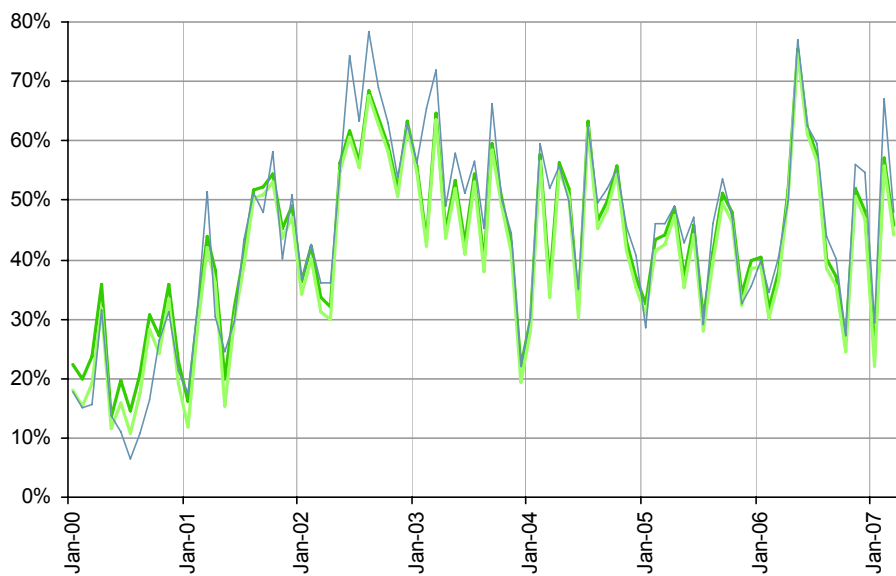

(b)

Source: Dresdner Kleinwort

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Fundamentals of index volatility, constituent volatility, correlation and dispersion

Exhibit 1.2.2 —24-month realised correlation measures of the Dow Jones EuroStoxx 50 index (2000—2007)

In the chart below, we compare the 'Proxy' and 'Average pair-wise' correlation measures with an alternative measure named 'Synthetic Standard', whose formula is given below. For this measure, we calculated: (i) realised index volatility using synthetic index values following (110) with constant weights and constituents throughout each 24-month period; (ii) realised constituent volatility as weighted arithmetic average of the realised volatilities of the constituent stocks.

Synthetic Standard: $\dfrac{\left(\sigma^{I}\right)^{2}-\varepsilon^{2}}{\left(\sum_{i=1}^{N} w_{i}\sigma^{S_{i}}\right)^{2}-\varepsilon^{2}}$.
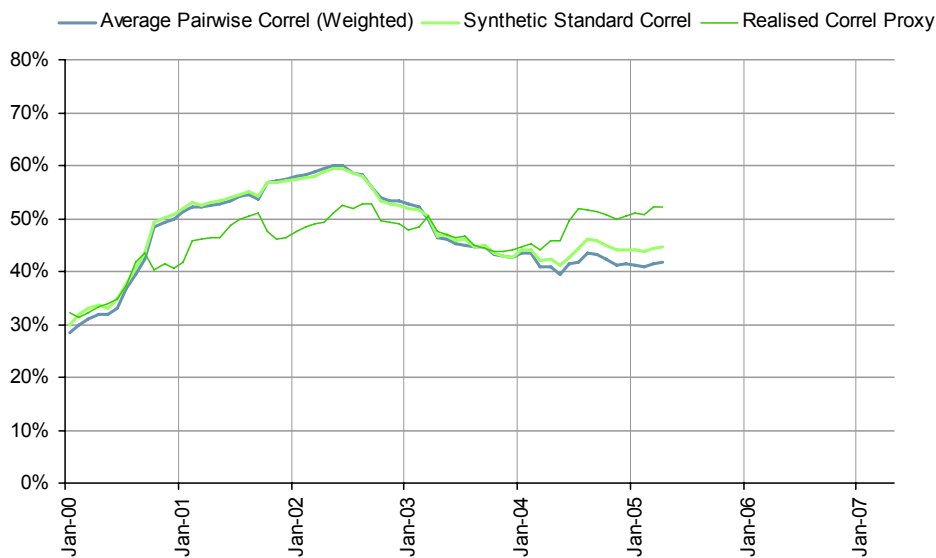
Source: Dresdner Kleinwort

1.3. Variance dispersion trades

One practical application of the proxy formulas is the so-called vega-neutral variance dispersion trade. Variance dispersion trades are spread trades between constituent variance and index variance, i.e. given a time period τ, the variance dispersion payoff is of the form:

$$D(\beta,\tau) \equiv \beta\left(\bar{\sigma}^{S}(\tau)\right)^{2} - \left(\sigma^{I}(\tau)\right)^{2}$$

where β is a positive constant called beta factor, leg ratio or spread ratio. The no-arbitrage price to enter into this trade at time t is: $D_{t}^{*}(\beta,\tau) \equiv \beta\left(\bar{\sigma}_{t}^{*S}(\tau)\right)^{2} - \left(\sigma_{t}^{*I}(\tau)\right)^{2}$. For ease of notation, we omit the time period τ as an argument in the rest of this report, except when placing particular emphasis.

In its standard form, $\beta = 1$ and we have $D(1) = d^{2}$. It must be noted that if $\beta < 1$ we can have $D_{t}^{*}(\beta) < 0$, and/or $D(\beta) < 0$. For the avoidance of doubt, the net exchange of cash flows at maturity is always determined by the differential amount $D(\beta) - D_{t_{0}}^{*}(\beta)$, where t_{0} is the trade date. If positive the dispersion seller pays the amount to the dispersion buyer, if negative the dispersion buyer pays the absolute amount to the dispersion seller.

The rationale for entering a variance dispersion trade is usually to sell or buy correlation through index variance, while hedging the unwanted volatility exposure through an offsetting position in constituent variance. Such trades have been particularly popular with hedge funds, lured by the large historical alpha between implied and realised correlation, as illustrated in Exhibit 1.3.1 for the Dow Jones EuroStoxx 50 index.

Fundamentals of index volatility, constituent volatility, correlation and dispersion

A crucial aspect of variance dispersion trades is the determination of the beta factor which controls the amount of constituent variance needed to isolate the correlation contained in index variance. Possible approaches include:

▶ Standard: $\beta = \dfrac{\sigma_{t_0}^{*I}}{\bar{\sigma}_{t_0}^{*S}}$. This case corresponds to the first variance dispersion trades. The rationale was that

aggregate vega[12] at time t_0 was nil: $\beta \left. \dfrac{\partial \left(\bar{\sigma}_t^{*S} \right)^2}{\partial \bar{\sigma}_t^{*S}} \right|_{t=t_0} - \left. \dfrac{\partial \left(\sigma_t^{*I} \right)^2}{\partial \sigma_t^{*I}} \right|_{t=t_0} = 2\beta\bar{\sigma}_{t_0}^{*S} - 2\sigma_{t_0}^{*I} = 0$. However, it soon

became clear to arbitrageurs that index and constituent volatility did not vary identically, which motivated more sophisticated choices for beta.

▶ Statistical. In this case, beta is chosen to minimise historical deviation between the variance dispersion payoff and an objective function such as $\rho_{t_0}^* - \rho$. Several practical studies of this approach can be found in broker-dealer reports.

▶ Fundamental: $\beta = \left(\dfrac{\sigma_{t_0}^{*I}}{\bar{\sigma}_{t_0}^{*S}} \right)^2 = \hat{\rho}_{t_0}^*$. The rationale is that we can rewrite the variance dispersion payoff as:

$D(\beta) = \left(\hat{\rho}_{t_0}^* - \hat{\rho} \right) \left(\bar{\sigma}^S \right)^2$, and that this approach has zero cost: $D_t^*(\beta) = 0$. The combination of these two

properties means that the *net* profit or loss of the variance dispersion trade is driven by the differential between implied and realised correlation, with realised constituent variance playing the role of a scaling factor. The fundamental approach is thus perfectly suited to capture the correlation alpha.

Dispersion trades weighted using the fundamental approach are named 'vega-neutral' because at time t_0 instantaneous moves in index volatility are perfectly compensated by moves in beta-weighted constituent volatility, provided implied correlation remains constant:

$$\left. \frac{\partial \left(\beta\bar{\sigma}_t^{*S} - \sigma_t^{*I} \right)^2}{\partial \bar{\sigma}_t^{*S}} \right|_{t=t_0} = 2\hat{\rho}_{t_0}^* \bar{\sigma}_{t_0}^{*S} - \left. \frac{\partial \left(\hat{\rho}_t^* \sigma_t^{*S} \right)^2}{\partial \sigma_t^{*S}} \right|_{t=t_0} = 0$$

For more details on vega-neutral dispersion trades, see e.g. Bossu-Gu (2004).

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[12] Vega is the sensitivity of a derivative contract, such as a variance swap, to changes in implied volatility. Aggregate vega is the sum of vegas of a portfolio of derivatives.

Fundamentals of index volatility, constituent volatility, correlation and dispersion

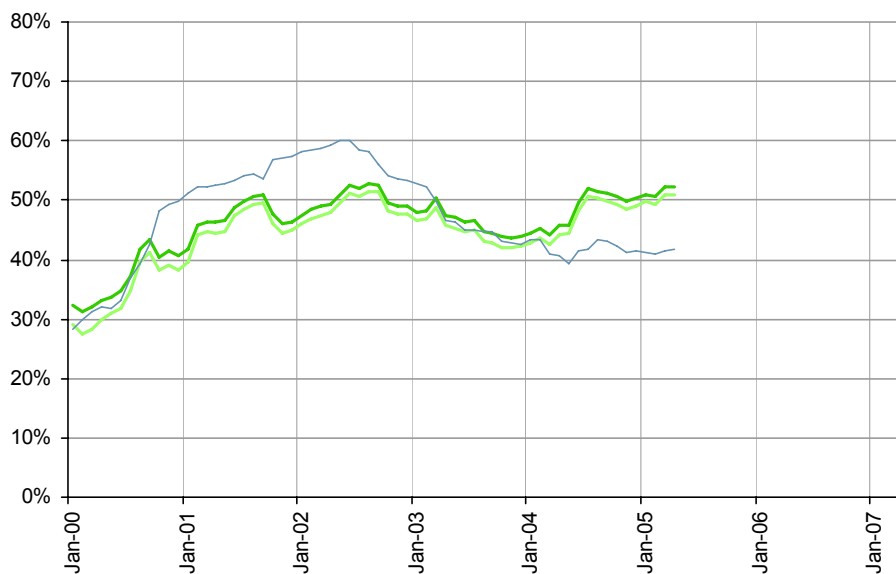**Exhibit 1.3.1 — 1-month and 12-month implied versus realised correlation of the Dow Jones EuroStoxx 50 index (2000—2007)**
In the two charts below, we report on each monthly listed expiry date the implied correlation derived from (a) 1-month or (b) 12-month at-the-money implied volatility levels together with realised correlation over the following (a) 1 or (b) 12 months, as well as their differential (alpha).

(a)

(b)
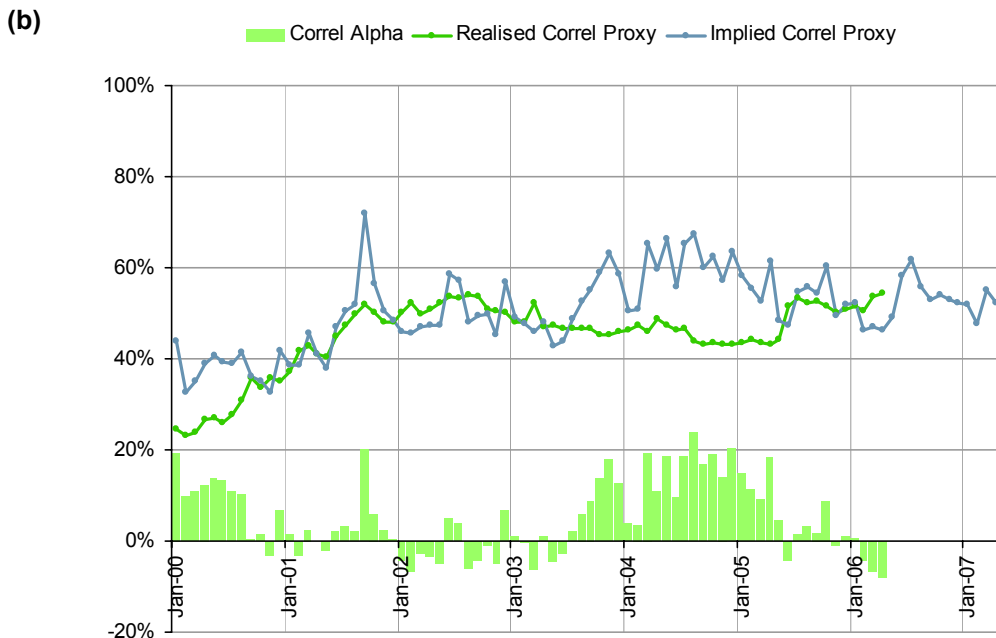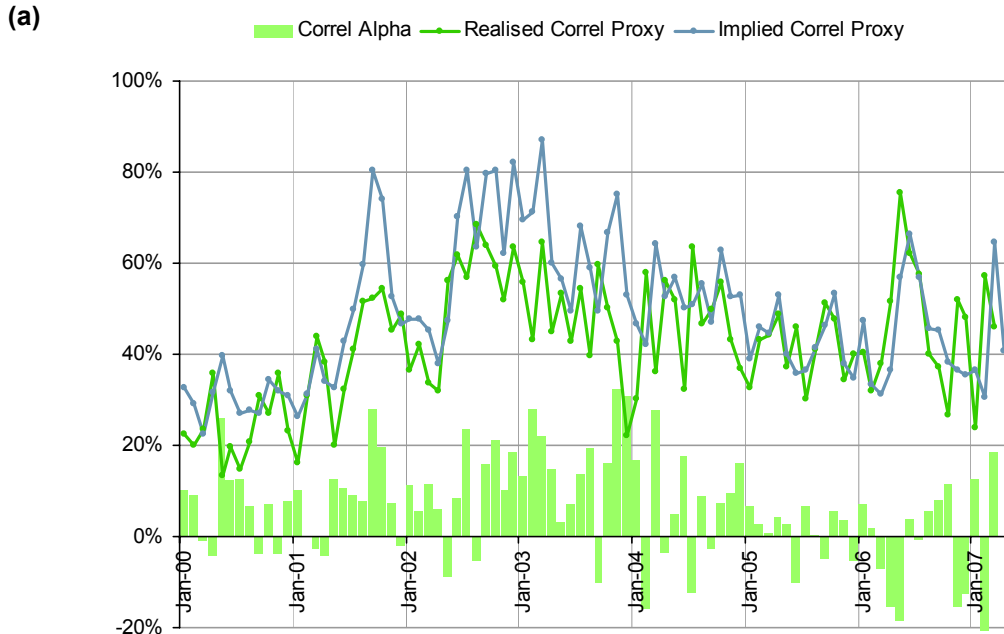
Source: Dresdner Kleinwort

Toy model for derivatives on realised variance

2.   Toy model for derivatives on realised variance

2.1.   Model framework

In this section we introduce a toy model for the fair value of derivative claims on realised variance which is a straightforward modification of the Black-Scholes model (1973). Our approach is similar to Duanmu (2004) or Friz-Gatheral (2005) and falls in the category of fixed-maturity variance models. More sophisticated approaches, such as Dupire (1992), Potter (2004), Carr-Sun (2005), or Buehler (2006) would make analytical formulas difficult or perhaps impossible to derive.

Our purpose is to extend this model in Section 3 in order to derive simple analytical formulas for a derivatives claim on realised correlation.

We consider a market on a single asset A where agents can buy or sell the asset's realised variance $\left(\sigma^A(\tau)\right)^2$ over a fixed time period τ = [0, T]. For $0 \le t \le T$, we denote v_t^* the corresponding price, which is paid at maturity T; in particular we have $v_0^* = \left(\sigma_0^{*A}(\tau)\right)^2$, and $v_T^* = \left(\sigma^A(\tau)\right)^2$. Generally, in the absence of arbitrage we must have for all t:

$$v_t^* = \frac{t}{T}\left(\sigma^A([0,t])\right)^2 + \frac{T-t}{T}\left(\sigma_t^{*A}([t,T])\right)^2$$

We specify the forward-neutral dynamics of v^* as follows:

$$dv_t^* = 2\omega\frac{T-t}{T}v_t^* dz_t^*$$

where ω is a positive 'volatility of volatility' parameter, and z^* is a standard Brownian motion under P*.

Hence, v^* is a geometric Brownian motion whose volatility parameter is time-dependent and linearly collapses to zero as maturity approaches[13]. Here we differ from Duanmu's approach, who additionally imposes a term structure of the volatility of volatility parameter of the form $\omega_t = \dfrac{\xi}{\sqrt{T-t}}$, resulting in a geometric Brownian motion v' whose volatility parameter decays with the square root of time:

$$dv_t' = 2\xi\frac{\sqrt{T-t}}{T}v_t' dz_t^*.$$

The advantage of Duanmu's approach is to give a more realistic time-dependence of the volatility of volatility parameter, as one can intuitively expect ω to be higher in the short term than in the long term[14]. The disadvantage is that it is an arbitrary choice, leading to a counter-intuitive result which we highlight at the end of the following section.

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[13] By Ito-Doeblin, the corresponding process for implied volatility $v^{*1/2}$ is also a geometric Brownian motion with dynamics of the form: $d\left(v_t^{*\frac{1}{2}}\right) = ...v_t^{*\frac{1}{2}}dt + \omega\frac{T-t}{T}v_t^{*\frac{1}{2}}dz_t^*$. This justifies our qualification of ω as a volatility of volatility parameter, as opposed to volatility of variance.

[14] Our empirical results in Appendices D and E corroborate this intuitive idea.

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Toy model for derivatives on realised variance

2.2.   Fair value of derivatives on realised variance. Volatility claims.

Within this framework, the fair value of any European derivative on realised variance v_T^* with payoff $f_T = f(v_T^*)$ is given as:

$$f_t = E_t^* \left(f(v_T^*) \right)$$

where *f* is a function satisfying the regularity conditions of the Ito-Doeblin theorem.

We must emphasise that for non-trivial functions *f*, the fair value f_t is model-dependent and relies on dynamically trading the asset's realised variance, which is why we do not distinguish it with an asterisk.

An important application is the determination of the fair value of the volatility claim[15] whose payoff is $\mathcal{V}_T \equiv \sqrt{v_T^*}$ (see Appendix B):

(**220**) $\quad \mathcal{V}_t = \sqrt{v_t^*} \exp \left[-\frac{1}{6} \omega^2 T \left(\frac{T-t}{T} \right)^3 \right]$

For *t* = 0, we find:

(**221**) $\quad \mathcal{V}_0 = \sqrt{v_0^*} \exp \left[-\frac{1}{6} \omega^2 T \right]$

In practice, $\mathcal{V}_0$ is known as the fair strike of a volatility swap, $\sqrt{v_0^*}$ is known as the fair strike of a variance swap, and the ratio between the two is known as the convexity adjustment[16]. We refer to these concepts as fair volatility, fair variance and quadratic adjustment, respectively. Readers should keep in mind that fair variance is the *square root* of implied variance.

A first-order expansion of (221) yields the following rule of thumb for the quadratic adjustment:

$$\frac{\sqrt{v_0^*}}{\mathcal{V}_0} - 1 \approx \frac{1}{6} \omega^2 T$$

Interestingly, this quadratic adjustment is insubstantial if we use for ω the typical implied volatility levels of 15 to 30% observed on stock index option markets. This should not be surprising since there is no reason to believe that volatility of volatility should be of same order as volatility. In fact, if we believe that fair volatility and variance should substantially differ, (221) suggests that the volatility of volatility parameter should be significantly higher than 30%.

(221) differs from Duanmu's result[17], who finds that the quadratic adjustment does not depend on maturity *T*. This counters the intuition that the longer the maturity, the higher the quadratic effect, as our empirical results in Appendices D and E also suggest up to 6-month maturity.

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[15] On over-the-counter markets, volatility claims are known as volatility swaps.

[16] The idea behind the convexity adjustment is that variance is convex in volatility: by Jensen's inequality, we must have $\sqrt{v_0^*} \geq \mathcal{V}_0$.

[17] Specifically, Duanmu states that $\mathcal{V}_0 = \sqrt{v_0'} \exp \left(-\frac{1}{4} \xi^2 \right)$, with *ξ* being the 1-year volatility of volatility.

Toy model for derivatives on realised variance

2.3. Parameter estimation

After Carr-Lee (2005) established a model-independent calculation of fair volatility under certain assumptions[18], Friz-Gatheral (2005) proposed to model terminal realised variance with a displaced log-normal distribution and calibrate the parameters to fair volatility and variance. This approach is sensible so long as one believes volatility claims can indeed be efficiently synthesised within the Carr-Lee framework.

In practice, however, volatility claims, known as volatility swaps on over-the-counter markets, continue to be illiquid instruments and their price is not observable. In Appendix D, we propose to estimate the volatility of volatility parameter ω for various maturities using a break-even historical analysis of the quadratic adjustment. Our empirical results for the Dow Jones EuroStoxx 50 index for the period 2000—2005 suggest that implied volatility of volatility should be of the order of 100% for short maturities, and of the order of 50% for longer maturities.

2.4. Model limitations

Notwithstanding the limitations typically attributed to the Black-Scholes model, the toy model cannot be straightforwardly extended to the entire variance curve. This is because the dynamics of the variance price process are geometric rather than arithmetic, producing inconsistencies across the curve[19].

Another limitation of the toy model is that it disregards the dynamics of the asset price process, and any joint dynamics between the asset price and its realised variance. It is unclear to us how these aspects would affect our results.

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[18] Specifically, Carr-Lee state that, assuming no correlation between the asset price process and its realised volatility, fair volatility is close to the implied volatility of an at-the-money-forward call or put. Carr-Lee also find a correlation-robust result involving Bessel functions.

[19] Recall that variance is additive: $\left(\sigma^A(\tau_1 \cup \tau_2) \right)^2 = \left(|\tau_1| + |\tau_2| \right)^{-1} \left[|\tau_1| \left(\sigma^A(\tau_1) \right)^2 + |\tau_2| \left(\sigma^A(\tau_2) \right)^2 \right]$, for any two disjoint time periods τ_1 and τ_2.

Rational pricing of equity correlation swaps

3. Rational pricing of equity correlation swaps

In this Section we extend the toy model to two factors[20] to represent (S, w, I), and derive an analytical formula for the fair value of a correlation claim with a payoff given as:

$$c_T \equiv \hat{\rho}(\tau) = \left(\frac{\sigma^I(\tau)}{\overline{\sigma}^S(\tau)} \right)^2$$

Note this definition differs from the standard payoff of the correlation swap trading over-the-counter:

$$\frac{\sum_{i<j} w_i w_j \rho^{S_i, S_j}(\tau)}{\sum_{i<j} w_i w_j}$$

where $\rho^{S_i, S_j}(\tau) \equiv |\tau|^{-1} (\sigma^{S_i} \sigma^{S_j})^{-1} \int_\tau (d \ln S_i)(d \ln S_j)$.

As shown in Exhibit 3.0.1 below, the historical difference between the above payoff and the correlation claim's payoff was close to zero on average for the Dow Jones EuroStoxx 50 index, using 1-month rolling time periods between 2000 and 2007; in 80% of the cases, the difference was comprised between -4.6 and +6.7 correlation points. This suggests that the fair value of a standard correlation swap should be close to that of the correlation claim, for short maturities.

Exhibit 3.0.1 — Payoff differential between a 1-month standard correlation swap (average pair-wise) and a 1-month correlation claim (proxy), for the Dow Jones EuroStoxx 50 index (2000—2007)
In the chart below, we report on each monthly listed expiry date the difference between the 'Average pair-wise' and 'Proxy' realised correlation measures, which coincides with the payoff differential between a 1-month standard correlation swap and a 1-month correlation claim. The calculation methodology is identical to Exhibit 1.2.1.
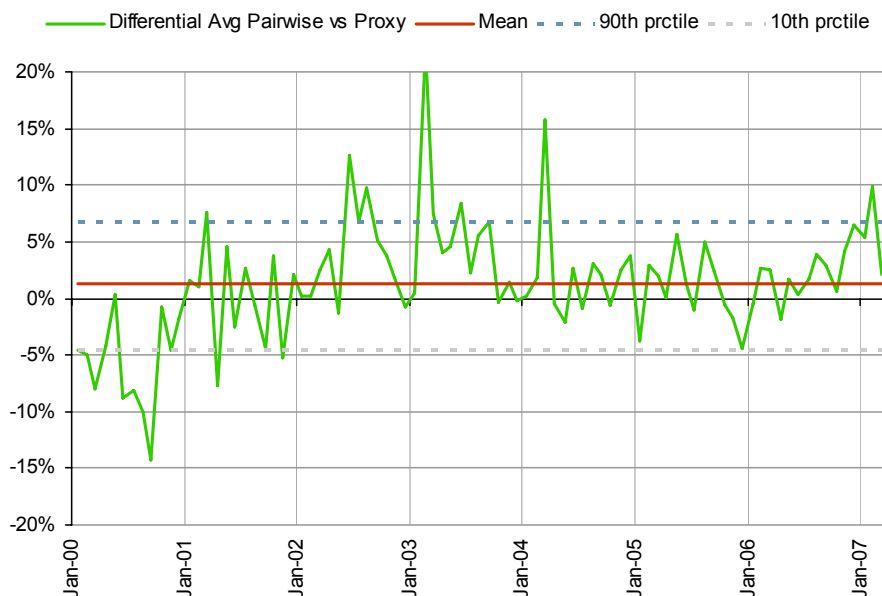
Source: Dresdner Kleinwort

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[20] Namely: index variance and constituent variance. The two-factor toy model must not be confused with two-factor models for the correlation between two assets, such as proposed by Dupire (1992).

Rational pricing of equity correlation swaps

3.1. Two-factor toy model

Given a fixed time period τ, denote v_t^{*I} the market price of realised index variance at time t, $v_t^{*S_i}$ the market price of the realised variance of stock S_i at time t, and define the no-arbitrage price of realised constituent variance at time t as: $\bar{v}_t^{*S} \equiv \sum_{i=1}^{N} w_i v_t^{*S_i}$.

We specify the forward-neutral dynamics of v^{*I} as follows:

$$dv_t^{*I} = 2\omega_I \frac{T-t}{T} v_t^{*I} dz_t^{*I}$$

where ω_I is a constant and z^{*I} is a standard Brownian motion under P*.

Similarly, we specify the dynamics of $\bar{v}^{*S}$ directly:

(310) $d\bar{v}_t^{*S} = 2\bar{\omega}_S \frac{T-t}{T} \bar{v}_t^{*S} dz_t^{*S}$

where $\bar{\omega}_S$ is a constant and z^{*S} is a standard Brownian motion under P*. It must be emphasized here that (310) is only an approximation of an *N*-factor approach with each v^{*S_i} following this type of dynamics. This is because an arithmetic average of log-normal variables is not necessarily log-normal.

Assuming that the instantaneous correlation between z^{*I} and z^{*S} is constant (i.e. $\left(dz_t^{*I}\right)\left(dz_t^{*S}\right) = \chi dt$), we can rewrite (**310**) as:

$$d\bar{v}_t^{*S} = 2\bar{\omega}_S \frac{T-t}{T} \bar{v}_t^{*S} \left[\chi dz_t^{*I} + \sqrt{1-\chi^2} dz_t^{*\perp}\right]$$

where $z_t^{*\perp} \equiv \dfrac{z_t^{*S} - \chi z_t^{*I}}{\sqrt{1-\chi^2}}$ is a standard Brownian motion under P*, orthogonal to z^{*I} by construction.

3.2. Fair value of the correlation claim

In this framework, the payoff of the correlation claim is a function of the two tradable assets v_t^{*I} and $\bar{v}_t^{*S}$:

$$c_T \equiv \frac{v_T^{*I}}{\bar{v}_T^{*S}} .$$

In Appendix C, we derive an analytical formula for the fair value of the correlation claim:

(320) $c_t \equiv E_t^*\left(c_T\right) = \dfrac{v_t^{*I}}{\bar{v}_t^{*S}} \exp\left[\dfrac{4}{3}\left(\bar{\omega}_S^2 - \chi\bar{\omega}_S\omega_I\right)T\left(\dfrac{T-t}{T}\right)^3\right]$

In other words, the fair value of the correlation claim is equal to the ratio of fair index variance to fair constituent variance, multiplied by an adjustment factor which depends on the volatility of index volatility, the volatility of constituent volatility, and the correlation between index and constituent volatilities.

In particular, at *t* = 0, we have:

(321) $\dfrac{\hat{\rho}_0^*}{c_0} = \exp\left(\dfrac{4}{3}\left(\chi\bar{\omega}_S\omega_I - \bar{\omega}_S^2\right)T\right)$

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A first-order expansion gives the rule of thumb:

$$\frac{\hat{\rho}_0^*}{c_0} - 1 \approx \frac{4}{3}\left(\chi\overline{\omega}_S\omega_I - \overline{\omega}_S^2\right)T$$

Here, we must emphasise that χ corresponds to the correlation between changes in index and constituent volatilities, *not* the correlation between absolute levels. Intuitively, the square of this parameter corresponds to the proportion of changes in index volatility explained by changes in constituent volatility (the rest being explained by changes in correlation.)

3.3. Parameter estimation

To estimate the volatility of constituent volatility parameter $\overline{\omega}_S$, we follow the same approach as for index volatility. Our results are given in Appendix F.

Estimating the correlation parameter χ is difficult because it requires to reconstitute the time series for v^{*I} and v^{*S}. In Appendix F, we use a break-even historical analysis to estimate χ, and we find values in the range 80% to 98%.

In Exhibit 3.3.1 below, we report our estimates for ω_I and $\overline{\omega}_S$, and calculate the fair correlation adjustment obtained in (321) using five hypothetical values for χ between 0.6 and 1. We can see that, for implied correlation to be higher than fair correlation, χ must be above 0.8. When this is the case, the fair correlation adjustment is close to 1 across all maturities.

These empirical results suggest that the fair value of a correlation claim should be close to implied correlation. Even if index and constituent volatilities were perfectly correlated, only extreme volatility of volatility parameter assumptions would result in a significant discrepancy between implied and fair correlations.

Exhibit 3.3.1 — Fair correlation adjustment (ratio of implied correlation to fair correlation), using theoretical volatility of volatility parameters, for various values of the instantaneous correlation between index and constituent volatilities χ

Maturity	Index theoretical volatility of volatility ω_I	Constituent theoretical volatility of volatility $\overline{\omega}_S$	Fair correlation adjustment (χ = 0.6)	Fair correlation adjustment (χ = 0.7)	Fair correlation adjustment (χ = 0.8)	Fair correlation adjustment (χ = 0.9)	Fair correlation adjustment (χ = 1)
1m	144.7%	123.4%	0.951	0.970	0.990	1.009	1.030
2m	122.6%	101.2%	0.940	0.966	0.993	1.021	1.049
3m	109.2%	88.9%	0.933	0.964	0.995	1.028	1.062
6m	86.5%	69.9%	0.920	0.957	0.997	1.038	1.081
12m	60.5%	54.1%	0.880	0.919	0.960	1.003	1.047
24m	41.5%	38.6%	0.869	0.906	0.946	0.987	1.031

Source: Dresdner Kleinwort

Dresdner Kleinwort

Rational pricing of equity correlation swaps

3.4.   Hedging strategy

We now analyse the hedging strategy corresponding to the fair value of the correlation claim given in (320). Our finding is that correlation swaps are replicated by dynamically trading vega-neutral variance dispersions.

The hedge ratios, or deltas, for index variance and constituent variance are given as:

$$\Delta_t^I = \frac{\partial c_t}{\partial v_t^{*I}} = \frac{c_t}{v_t^{*I}}$$

$$\Delta_t^S = \frac{\partial c_t}{\partial \bar{v}_t^{*S}} = -\frac{c_t}{\bar{v}_t^{*S}}$$

Hence, a long position in a correlation claim is hedged with a short position in index variance and a long position in constituent variance, i.e. a long variance dispersion trade. The beta factor between the two legs of the dispersion trade is given as:

$$\beta_t = \left| \frac{\Delta_t^S}{\Delta_t^I} \right| = \frac{v_t^{*I}}{\bar{v}_t^{*S}}$$

In particular, at time t = 0, we have $\beta_0 = \hat{\rho}_0^*$: the initial hedge is a vega-neutral dispersion trade. It is easy to see that subsequent dynamic hedges aim to maintain vega-neutrality until maturity. Note that the hedge has zero cost, and that β_t generalises $\hat{\rho}_t^*$ to time periods starting in the past.

3.5.   Model limitations

In addition to the limitations of the one-factor case, the two-factor toy model is not entirely arbitrage-free, as it allows for $v_t^{*I} > v_t^{*S}$, i.e. $\hat{\rho}_t^* > 1$. Our numerical results in Appendix G tend to indicate that, for $\chi > 80\%$ and typical values for other model parameters, the probability of the *terminal* realised correlation c_T being above 1 would be less than 5%. This is also confirmed in Exhibit 3.5.1 below where we used the theoretical volatility of volatility estimates found in Appendices D and E and a value of 50% for the initial implied correlation $\hat{\rho}_0^*$.

Here, we seem to have a trade-off between model simplicity and accuracy. Because index and constituent variances are traded assets, we cannot introduce mean reversion or any other type of constraint on their drift under the forward-neutral measure P*. One solution could be to make the correlation of volatilities parameter χ non-constant, so as to further reduce the probability of an arbitrage. Another, more ambitious solution might be to develop a large-factor model consistent with option and variance prices on the index and each constituent stock[21].

Another limitation of the toy model is that it assumes static weights and constituent stocks. While this assumption seems reasonable for short time periods, it is likely to affect results for longer time periods.

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[21] A step in this direction can be found in Driessen et al. (2005), who derive endogenous dynamics for index variance based on the dynamics of constituent variances and an instantaneous correlation of stock returns process of the Wright-Fisher type. However, the model of Driessen et al. is for variances and correlations of constant rolling maturity, which are non-tradable assets. It is unclear to us whether their results can be extended to the fixed maturity case, with forward neutral drifts.

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Exhibit 3.5.1 — Probability of $c_T > 1$ in function of maturity T, for various values of the correlation between index and constituent volatilities parameter χ, and initial implied correlation $\hat{\rho}_0^* = 0.5$

In the chart below, we report for each maturity the implied probability of the terminal realised correlation c_T being above 1 assuming an initial implied correlation of 50%. To construct each curve, we applied the analytical formula found in Appendix G with a given value for χ and the theoretical estimates found in Appendices D and E for ω's.
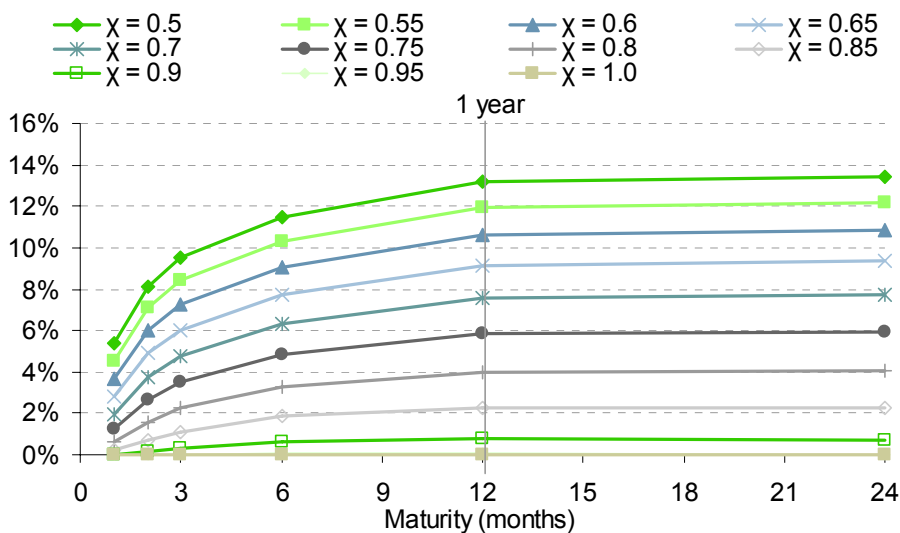
Source: Dresdner Kleinwort

Further research

4.   Further research

Despite its limitations, our approach is, to our knowledge, the first of its kind to establish that correlation swaps on the constituents of a stock index can be replicated by dynamically trading variance dispersions, and that their fair value is straightforwardly related to implied correlation. In fact, using a parameter estimation methodology which relies on few historical factors, we obtain numerical results supporting the intuitive idea that the fair value of a correlation swap should be close to implied correlation. Dynamic arbitrage opportunities may therefore exist whenever the market price of correlation swaps substantially differs from implied correlation.

Further research is now needed:

► On the fundamental side, the toy model needs to be refined to be made entirely arbitrage-free.

► On the practical side, the toy model needs to be extended in order to calculate the fair value of other correlation measures, for instance the canonical or average pair-wise measures. Additionally, allowing for free-float weights and changes in index composition would render the model closer to index calculation practices.

► On the numerical side, more sophisticated parameter estimations, over longer historical periods and in other markets, would be extremely valuable.

We believe our approach constitutes a first step towards a consistent pricing theory of statistical derivatives (i.e. derivatives on realised variance and correlation). The toy model provides us with elementary analytical formulas and rules of thumbs. We also see two other research areas which may benefit from our results: the pricing and hedging of exotic derivatives on multiple stocks (in particular the modelling inclusion of correlation skew), and the stochastic modelling of volatility and correlation.

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[22] When referring to a working paper, we indicate the year of the first known version after the author's name, followed by the title and date of the latest known version.

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Appendices

6.   Appendices

A — Proxy formulas for realised and implied correlations

In this appendix, we render the dependence on *τ* implicit and the dependence on *N* explicit in the notations introduced in Section 1.1. Define the pair-wise correlation coefficient between stocks S_i and S_j as:

$$\rho^{S_i,S_j} \equiv |\tau|^{-1}\left(\sigma^{S_i}\sigma^{Sj}\right)^{-1}\int_\tau (d\ln S_i)(d\ln S_j)$$

Write:

$$\left(\sigma^I(N)\right)^2 = \sum_{i\neq j} w_i w_j \sigma^{S_i}\sigma^{S_j}\rho^{S_i,S_j} + \left(\varepsilon(N)\right)^2$$

Define:

$$\left.\mathrm{P}\right|_{min}^{(N)} \equiv \min_{i,j}\left|\rho^{S_i,S_j}\right|, \ \sigma_{min}^{(N)} \equiv \min_i \sigma^{S_i}, \ \sigma_{max}^{(N)} \equiv \max_i \sigma^{S_i}, \ w_{min}^{(N)} \equiv \min_i w_i, \ w_{max}^{(N)} \equiv \max_i w_i$$

Assuming $\sigma_{min}^{(N)} > 0$ and $\left.\mathrm{P}\right|_{min}^{(N)} > 0$, we have:

(**A1**) $0 \leq \dfrac{\left(\varepsilon(N)\right)^2}{\left(\bar{\sigma}^S(N)\right)^2} \leq \dfrac{\left(\varepsilon(N)\right)^2}{\left(\sigma^I(N)\right)^2} \leq \dfrac{N\left(w_{max}^{(N)}\sigma_{max}^{(N)}\right)^2}{N^2 \left.\mathrm{P}\right|_{min}^{(N)}\left(w_{min}^{(N)}\sigma_{min}^{(N)}\right)^2} = \dfrac{1}{N}\cdot\dfrac{\left(w_{max}^{(N)}\right)^2}{\left.\mathrm{P}\right|_{min}^{(N)}\left(w_{min}^{(N)}\right)^2}\cdot\left(\dfrac{\sigma_{max}^{(N)}}{\sigma_{min}^{(N)}}\right)^2$

If we further assume that (a) all realised volatilities never degenerate towards zero nor explode towards infinity, and (b) all pair-wise correlations never degenerate towards zero, we obtain the reduced convergence condition:

$$\frac{w_{max}^{(N)}}{w_{min}^{(N)}} = o\left(\sqrt{N}\right)$$

Combined with inequality (A1), this condition ensures:

$$\begin{cases}\left(\varepsilon(N)\right)^2 = o\left(\left(\sigma^I(N)\right)^2\right) \\[2mm] \left(\varepsilon(N)\right)^2 = o\left(\left(\bar{\sigma}^S(N)\right)^2\right)\end{cases}$$

whence:

$$\rho(\tau)\xrightarrow[N\to+\infty]{}\left(\frac{\sigma^I(\tau)}{\bar{\sigma}^S(\tau)}\right)\equiv\hat{\rho}(\tau)$$

This condition is also sufficient to ensure the convergence of implied correlation:

$$\rho_t^*(\tau)\xrightarrow[N\to+\infty]{}\left(\frac{\sigma_t^{*I}(\tau)}{\bar{\sigma}_t^{*S}(\tau)}\right)\equiv\hat{\rho}_t^*(\tau)$$

Note that we can drop assumption (b) and obtain the relaxed convergence condition:

$$\frac{\left(w_{max}^{(N)}\right)^2}{\left.\mathrm{P}\right|_{min}^{(N)}\left(w_{min}^{(N)}\right)^2} = o(N)$$

It is clear that this condition ensures the convergence of realised correlation. We do not know if it is sufficient for implied correlation without making the additional market assumption that one can trade $\left.\mathrm{P}\right|_{min}^{(N)}$.

Appendices

B — Analytical formula for the volatility claim within the toy model

Applying the Ito-Doeblin theorem, we can write the diffusion equation for ln v:

$$d \ln(v_t^*) = -2\omega^2 \left(\frac{T-t}{T} \right)^2 + 2\omega \frac{T-t}{T} dz_t^*$$

Equivalently:

$$v_T^* = v_t^* \exp\left(-2\frac{\omega^2}{T^2} \int_t^T (T-s)^2 ds + 2\frac{\omega}{T} \int_t^T (T-s) dz_s^* \right)$$

Calculating the first integral explicitly:

(**B1**) $\quad v_T^* = v_t^* \exp\left[-\frac{2}{3} \omega^2 T \left(\frac{T-t}{T} \right)^3 + 2\frac{\omega}{T} \int_t^T (T-s) dz_s^* \right]$

Taking the square root and then the conditional expectation of both sides of (B1), we can write that the price of the volatility claim is given as:

(**B2**) $\quad \mathcal{V}_t = \sqrt{v_t^*} \exp\left(-\frac{1}{3} \omega^2 T \left(\frac{T-t}{T} \right)^3 \right) E_t^* \left[\exp\left(\frac{\omega}{T} \int_t^T (T-s) dz_s^* \right) \right]$

The stochastic integral $\int_t^T (T-s) dz_s^*$ has a conditional normal distribution with respect to F_t, with zero mean and standard deviation $\frac{1}{\sqrt{3}}(T-t)^{3/2}$. Thus:

$$E_t^* \left[\exp\left(\frac{\omega}{T} \int_t^T (T-s) dz_s^* \right) \right] = \exp\left(\frac{\omega^2}{2T^2} \int_t^T (T-s)^2 ds \right) = \exp\left(\frac{1}{6} \omega^2 T \left(\frac{T-t}{T} \right)^3 \right)$$

Substituting this result in (B2) we obtain:

$$\mathcal{V}_t = \sqrt{v_t^*} \exp\left(-\frac{1}{6} \omega^2 T \left(\frac{T-t}{T} \right)^3 \right)$$

Appendices

C — Analytical formula for c_t within the 2-factor toy model

Applying the Ito-Doeblin theorem on $\ln \dfrac{v^{*I}}{\bar{v}^{*S}}$, we find:

$$d\ln \frac{v_t^{*I}}{\bar{v}_t^{*S}} = 2\left(\bar{\omega}_S^2 - \omega_I^2\right)\left(\frac{T-t}{T}\right)^2 + 2(\omega_I - \chi\bar{\omega}_S)\frac{T-t}{T}dz_t^{*I} - 2\bar{\omega}_S\sqrt{1-\chi^2}\frac{T-t}{T}dz_t^{*\perp}$$

Equivalently:

$$(\textbf{C1})\ \ c_T = \frac{v_t^{*I}}{\bar{v}_t^{*S}}\exp\left[\frac{2}{3}\left(\bar{\omega}_S^2 - \omega_I^2\right)T\left(\frac{T-t}{T}\right)^3 + 2\frac{\omega_I - \chi\bar{\omega}_S}{T}\int_t^T (T-s)dz_s^{*I} - 2\frac{\bar{\omega}_S\sqrt{1-\chi^2}}{T}\int_t^T (T-s)dz_s^{*\perp}\right]$$

Taking conditional expectations under P* yields:

$$c_t = \frac{v_t^{*I}}{\bar{v}_t^{*S}}\exp\left[\frac{2}{3}\left(\bar{\omega}_S^2 - \omega_I^2 + (\omega_I - \chi\bar{\omega}_S)^2 + \bar{\omega}_S^2(1-\chi^2)\right)T\left(\frac{T-t}{T}\right)^3\right]$$

Expanding the squares and simplifying, we obtain:

$$c_t = \frac{v_t^{*I}}{\bar{v}_t^{*S}}\exp\left[\frac{4}{3}\left(\bar{\omega}_S^2 - \chi\bar{\omega}_S\omega_I\right)T\left(\frac{T-t}{T}\right)^3\right]$$

D — Estimation of the volatility of index volatility parameter

When historical prices for implied variance are available, as is the case with the major stock indexes such as the Dow Jones EuroStoxx50, several estimators can be constructed for the volatility of volatility parameter ω. The problem here is that ω does not correspond to the volatility of realised volatility, nor the volatility of implied volatility; ω corresponds to the volatility of the variance price process v_t^* over its lifetime. To get around this problem, we propose to estimate ω by isolating the quadratic effect of variance versus volatility.

Inverting (221), we can write:

$$\omega = \sqrt{\frac{6}{T}\ln\Gamma}$$

where Γ is the ratio of fair variance to fair volatility, respectively denoted $\sqrt{v_0^*}$ and $\mathcal{V}_0$ in (221). Note that if time series of fair variance and volatility were available, we could back out an implied volatility of volatility parameter ω for each historical date and analyse its statistics. However, in practice, volatility swaps are much less liquid instruments compared to variance swaps, precisely due to the absence of market consensus on their fair value. Our methodology is to determine ω such that historical spread trades between variance and volatility break even on average.

Given a sequence of M time periods $(\tau_m)_{m=1..M}$, define:

$$\hat{\Gamma} \equiv \left[1 - \frac{1}{2M}\sum_{m=1}^{M}\left(\frac{R_m - K_m}{K_m}\right)^2\right]^{-1}$$

where $R_m \equiv \sigma^I(\tau_m)$ denotes realised index volatility over τ_m, and $K_m \equiv \sigma_{\inf \tau_m}^{*I}(\tau_m)$ denotes fair variance at the start of time period τ_m.

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From an economic point of view, $\hat{\Gamma}$ corresponds to the historical quadratic adjustment to be used so that an arbitrageur repeating normalised spread trades between variance and volatility would break even on average. To see this clearly, assume that for each historical time period ordinal m, future realised volatility over τ_m trades at fair variance K_m divided by a constant quadratic adjustment factor γ. Buying $\dfrac{1}{2K_m^2}$ units of variance and selling $\dfrac{1}{K_m}$ units of volatility, and repeating the trade for all historical dates, the total profit or loss is:

$$p/l = \sum_{m=1}^{M}\left[\frac{R_m^2}{2K_m^2} - \frac{1}{2} - \left(\frac{R_m}{K_m} - \frac{1}{\gamma}\right)\right] = \sum_{m=1}^{M}\left[\frac{1}{2}\left(\frac{R_m - K_m}{K_m}\right)^2 - \left(1 - \frac{1}{\gamma}\right)\right]$$

Assuming p/l = 0 and solving for γ, we find $\gamma = \hat{\Gamma}$.

We applied this methodology on a monthly basis, for 1-, 2-, 3-, 6-, 12- and 24-month listed expiries, using at-the-money implied volatility levels as a proxy for fair variance. Exhibit D1 below shows the results obtained on the Dow Jones EuroStoxx 50 for the period 2000—2005 (64 data points). We can see that the break-even quadratic adjustment increases with the length of the time period up to 6 months and then remains stable, while the corresponding theoretical volatility of volatility decreases.

We must emphasise that this analysis is fairly coarse due to the small number of data points, and the significant changes in market conditions within the historical period[23]. As such, the figures obtained should not be seen as any form of recommended parameter values for trading purposes; they might, however, form the basis for sensible values in a risk management context.

For completeness, we mention two other possible methodologies which require the reconstitution of the price process $\left(v_t^*(\tau_m)\right)_{t\in\tau_m}$ for each time period τ_m:

► Realised volatility of volatility: $\hat{\omega}_m \equiv \dfrac{3}{4T}\displaystyle\int_{t\in\tau_m}\left(d\ln v_t^*(\tau_m)\right)^2$

► Break-even volatility of volatility: $\hat{\omega}_m$ is the solution to the break-even delta-hedging p/l equation:

$$\int_{t\in\tau_m}\left(\frac{\partial^2 \mathbf{V}}{\partial v^2}(t, v_t^*(\tau_m), \hat{\omega}_m, \tau_m)\right)v_t^*(\tau_m)^2\left[\left(\frac{dv_t^*(\tau_m)}{v_t^*(\tau_m)}\right)^2 - 4\hat{\omega}_m^2\left(\frac{\sup\tau_m - t}{|\tau_m|}\right)^2\right]dt = 0$$

where: $\mathbf{V}(t, v, \omega, \tau) \equiv \sqrt{v}\exp\left[-\frac{1}{6}\omega^2|\tau|\left(\frac{\sup\tau - t}{|\tau|}\right)^3\right]$.

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[23] Internet bubble in 2000, September 2001 terror attacks, bear stock markets in 2002, war in Iraq in 2003, stock markets recovery since 2004.

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Exhibit D1 — Break-even quadratic adjustment and corresponding theoretical volatility of volatility for the Dow Jones EuroStoxx 50 index (2000—2005)

In the chart below, we report for each maturity the values of $\hat{\Gamma}$ and $\hat{\omega}$ observed on the Dow Jones EuroStoxx 50 index between 2000 and 2005, corresponding to 64 periods starting on a monthly listed expiry date. The calculation methodology of realised and implied volatilities is identical to Exhibit 1.2.1.
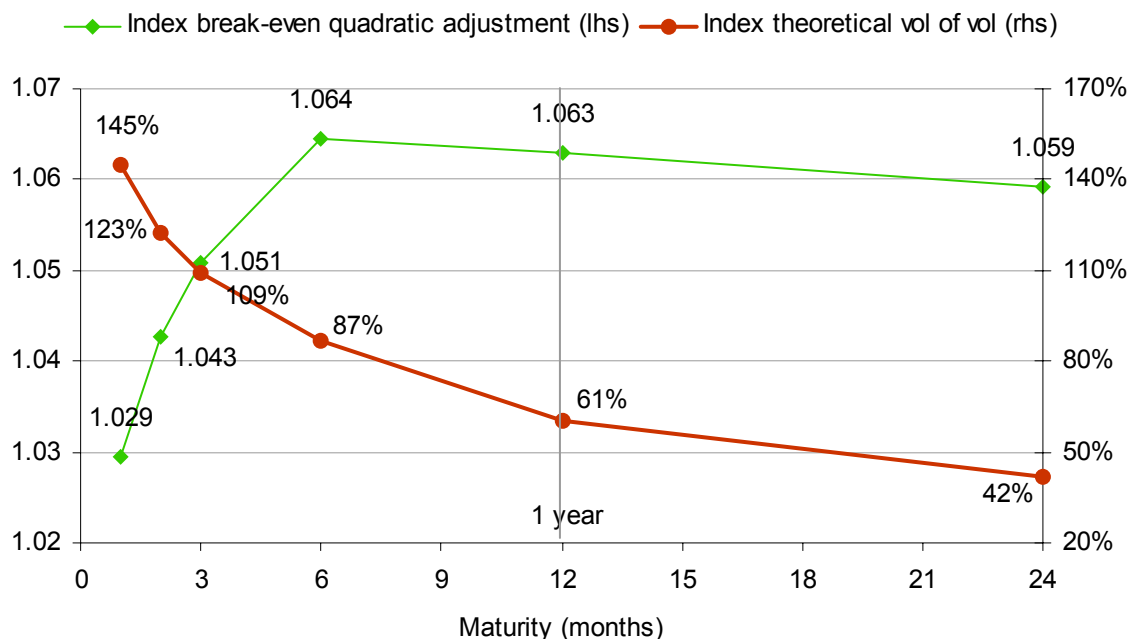
Source: Dresdner Kleinwort

E — Estimation of the volatility of constituent volatility parameter

We apply the same methodology as in Appendix D to estimate the volatility of constituent volatility parameter $\overline{\omega}_S$.

For an index such as the Dow Jones EuroStoxx 50, the difficulty here is to obtain reliable implied volatility surfaces for each of the 50 constituents. The amount of data mining involved in such an operation can be considerable; we limited ourselves to estimating historical at-the-money implied volatility levels on a monthly basis for 1-, 2-, 3-, 6-, 12- and 24-month listed expiries.

Exhibit E1 below shows the results we obtained for the period 2000—2005, based on 64 monthly data points. We can see that the break-even quadratic adjustment increases with the maturity, while the corresponding theoretical volatility of constituent volatility decreases. Compared with Exhibit D1, both figures are systematically lower than for the index. This should not be surprising: in any framework where constituent volatility and correlation have positive covariance, we have: $Var\left(\ln \sigma^I\right) = Var\left(\ln \overline{\sigma}^S + \frac{1}{2}\ln \hat{\rho}\right) \geq Var\left(\ln \overline{\sigma}^S\right)$.

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Exhibit E1 — Break-even quadratic adjustment and volatility of volatility of the constituents of the Dow Jones EuroStoxx 50 index for the period 2000—2005

In the chart below, we report for each maturity the values of $\hat{\Gamma}$ and $\hat{\omega}$ observed on the constituent stocks of the Dow Jones EuroStoxx 50 index between 2000 and 2005, corresponding to 64 periods starting on a monthly listed expiry date. The calculation methodology of realised and implied volatilities is identical to Exhibit 1.2.1.
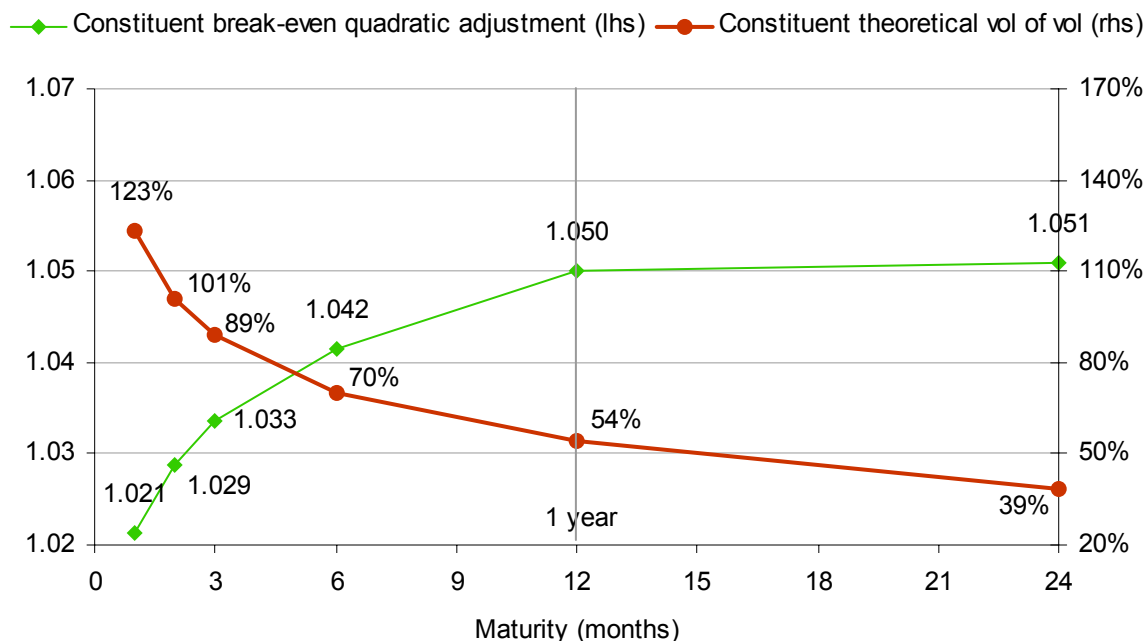
Source: Dresdner Kleinwort

F — Estimation of the correlation parameter between index and constituent volatilities

Inverting (321), we have:

$$\chi = \frac{\overline{\omega}_S^2 + \dfrac{3}{4T}\ln\Lambda}{\omega_I \overline{\omega}_S}$$

where Λ denotes the ratio of implied correlation to fair correlation, respectively denoted $\hat{\rho}_0^*$ and c_0 in (321).

Similarly to the situation described in Appendix D, if time series of fair correlation and implied correlation were available, we could back out an implied correlation of volatilities parameter χ for each historical date and analyse its statistics. In practice, correlation swaps are illiquid instruments and interbank quotes are infrequent.

Our methodology is to estimate χ using a break-even analysis of the fair to implied correlation adjustment. Given a sequence of M time periods $(\tau_m)_{m=1..M}$, define:

$$\hat{\Lambda} \equiv \left(\sum_{m=1}^{M} IC_m \left[\frac{RC_m}{IC_m} + \left(1 - \frac{RC_m}{IC_m}\right)\left(\frac{R_m}{K_m}\right)^2 \right] \middle/ \sum_{m=1}^{M} IC_m \right)^{-1}$$

where $R_m \equiv \overline{\sigma}^S(\tau_m)$ is the realised constituent volatility over τ_m, $K_m \equiv \overline{\sigma}^{*S}_{\inf \tau_m}(\tau_m)$ is the corresponding implied volatility at the start of time period τ_m, $RC_m \equiv \hat{\rho}(\tau_m)$ is the realised correlation over τ_m, and $IC_m \equiv \hat{\rho}^*_{\inf \tau_m}(\tau_m)$ is the corresponding implied volatility at the start of time period τ_m.

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Appendices

From an economic point of view, $\hat{\Lambda}$ corresponds to the historical adjustment to be used so that an arbitrageur repeating normalised spread trades between variance dispersion and correlation swaps would break even on average. To see this clearly, assume that for each historical time period ordinal m, future realised correlation over τ_m trades at implied correlation IC_m divided by a constant adjustment factor λ. Selling $\dfrac{1}{K_m^2}$ units of a vega-neutral dispersion[24] and selling 1 unit of correlation, and repeating the trade for all historical dates, the total profit or loss is:

$$p/l = \sum_{m=1}^{M}\left[\left(\frac{R_m}{K_m}\right)^2 (RC_m - IC_m) - \left(RC_m - \frac{IC_m}{\lambda}\right)\right] = \sum_{m=1}^{M}\left[RC_m\left(\left(\frac{R_m}{K_m}\right)^2 - 1\right) - IC_m\left(\left(\frac{R_m}{K_m}\right)^2 - \frac{1}{\lambda}\right)\right]$$

Assuming p/l = 0 and solving for λ, we find $\lambda = \hat{\Lambda}$.

We applied this methodology on a monthly basis, for 1-, 2-, 3-, 6-, 12- and 24-month listed expiries, using at-the-money implied volatilities to calculate implied correlation. Exhibit F1 below show the results obtained on the Dow Jones EuroStoxx 50 for the period 2000—2005 (64 data points). We can see that the break-even adjustment factor $\hat{\Lambda}$ takes values between 0.991 and 1.041 while the corresponding theoretical correlation parameter χ increases from 80.7% to 98.0%. We can also notice that, contrary to results in Appendices D and E for the break-even quadratic factor, the term structure of the break-even adjustment factor $\hat{\Lambda}$ is non-increasing.

Exhibit F1 — Break-even adjustment factor and theoretical correlation between index and constituent volatilities, for Dow Jones EuroStoxx 50 (2000—2005)

In table (a) below, we report for each maturity the values of $\hat{\Lambda}$, ω's and χ observed on the Dow Jones EuroStoxx 50 index and its constituents between 2000 and 2005, corresponding to 64 periods starting on a monthly listed expiry date. Chart (b) plots the term structure of $\hat{\Lambda}$, and χ only.

(a)

Maturity	Break-even adjustment factor $\hat{\Lambda}$	Index theoretical volatility of volatility ω_I	Constituent theoretical volatility of volatility $\overline{\omega}_S$	Theoretical correlation of volatilities χ
1m	0.991	144.7%	123.4%	80.7%
2m	1.013	122.6%	101.2%	87.2%
3m	1.027	109.2%	88.9%	89.8%
6m	1.041	86.5%	69.9%	90.8%
12m	1.018	60.5%	54.1%	93.5%
24m	1.022	41.5%	38.6%	98.0%

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[24] Recall that a short variance dispersion position is long correlation.

Dresdner Kleinwort

Appendices

(b)
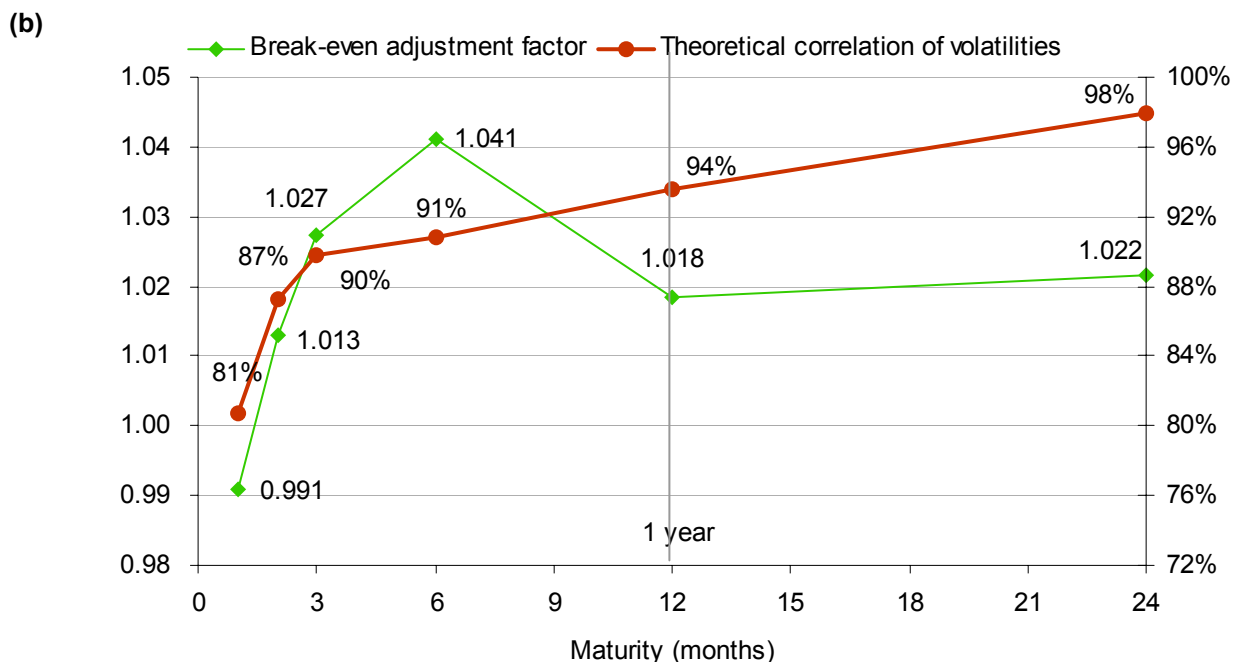
Source: Dresdner Kleinwort

G — Probability of $c_T > 1$ within the two-factor toy model

From the particularisation of (C1) at $t = 0$, we can write:

$$P^*\left(\{c_T > 1\}\right) = P^*\left(\left\{-2\frac{\omega_I - \chi\overline{\omega}_S}{T}\int_0^T (T-s)dz_s^{*I} + 2\frac{\overline{\omega}_S\sqrt{1-\chi^2}}{T}\int_0^T (T-s)dz_s^{*\perp} < \ln\hat{\rho}_0^* + \frac{2}{3}\left(\overline{\omega}_S^2 - \omega_I^2\right)T\right\}\right)$$

The two stochastic integrals in the above expression being independent normals with zero mean and variance $\frac{4}{3}T$, we obtain after simplifying terms:

$$P^*\left(\{c_T > 1\}\right) = N\left(\frac{\ln\hat{\rho}_0^* + \frac{2}{3}\left(\overline{\omega}_S^2 - \omega_I^2\right)T}{\frac{2\sqrt{T}}{\sqrt{3}}\sqrt{\omega_I^2 - 2\chi\omega_I\overline{\omega}_S + \overline{\omega}_S^2}}\right)$$
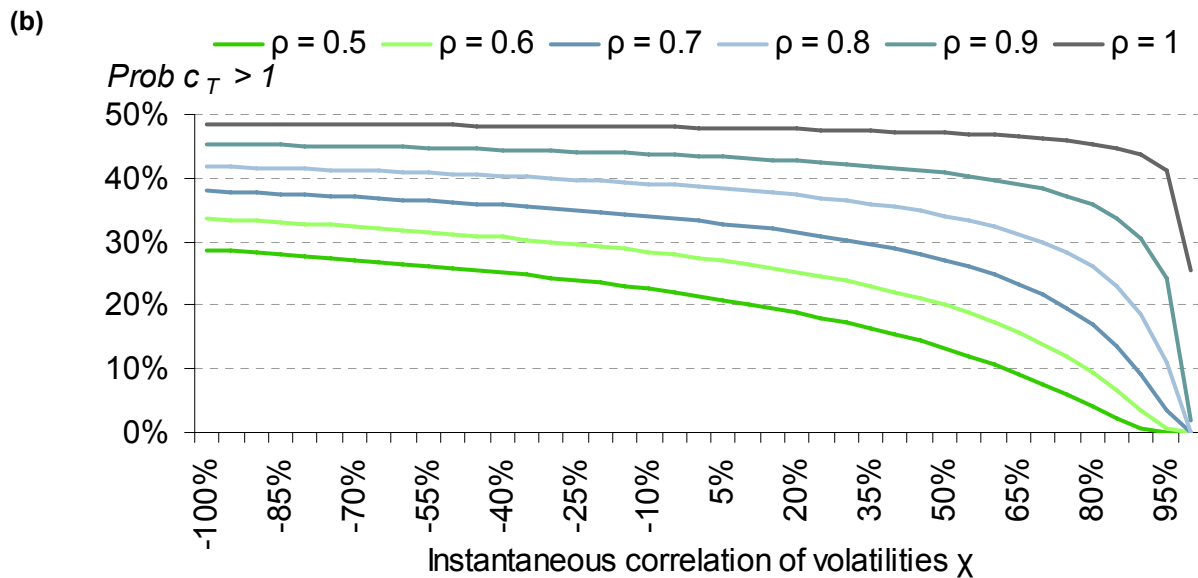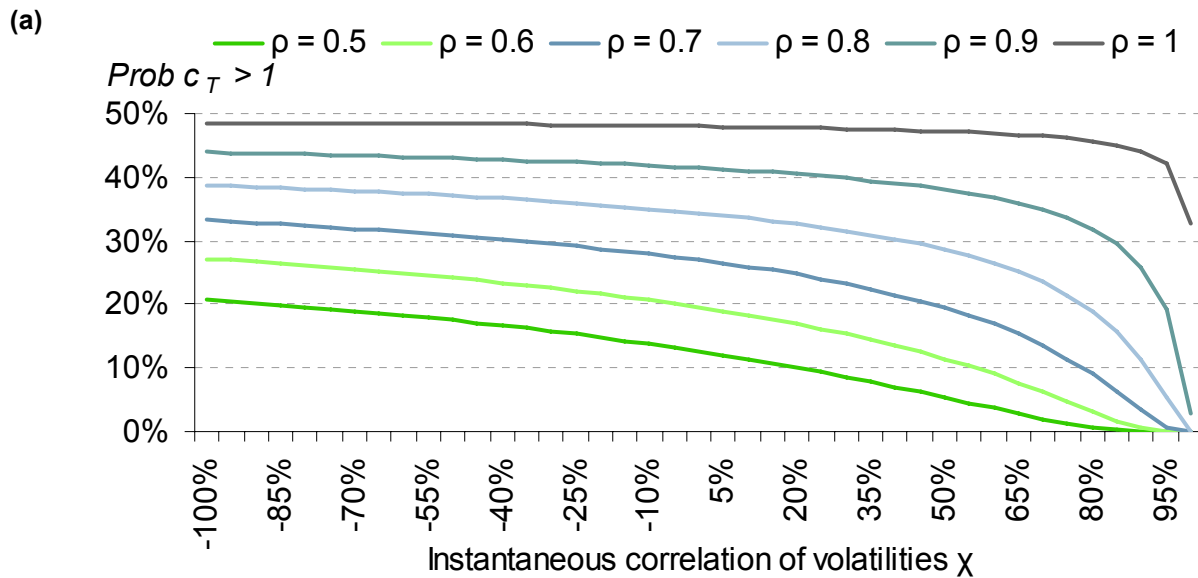
where $N(.)$ denotes the cumulative distribution of a standard normal.

We calculated this value for 1-month and 12-month maturities, using the theoretical volatility of volatility parameters found in Appendices D and E, for χ between -1 and 1 and $\hat{\rho}_0^*$ between 50% and 100%. Results are shown in Exhibit G1. We can see that the probability increases with $\hat{\rho}_0^*$ and decreases with χ.

Appendices

Exhibit G1 — Probability of $c_T > 1$ for $T = 1/12$, in function of the instantaneous correlation between index and constituent volatilities χ, for various values of the initial implied correlation $\hat{\rho}_0^*$.

In the charts below, we report the value of $P^*(\{c_T > 1\})$ for (a) 1-month and (b) 12-month maturity, in function of χ, using the values of ω's found in Appendices D and E. Each curve corresponds to a given value of $\hat{\rho}_0^*$.

(a)

(b)

Source: Dresdner Kleinwort

Dresdner Kleinwort